PACKING 3-VERTEX PATHS IN CLAW-FREE GRAPHS

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Abstract

A Λ-*factor* of a graph G is a spanning subgraph of G whose every component is a 3-vertex path. Let $v(G)$ be the number of vertices of G. A graph is *claw-free* if it does not have a subgraph isomorphic to $K_{1,3}$. Our results include the following. Let G be a 3-connected claw-free graph, $x \in V(G)$, $e = xy \in E(G)$, and L a 3-vertex path in G. Then $(c1)$ if $v(G) = 0 \bmod 3$, then G has a Λ-factor containing (avoiding) e, $(c2)$ if $v(G) = 1 \bmod 3$, then $G - x$ has a Λ-factor, $(c3)$ if $v(G) = 2 \bmod 3$, then $G - \{x, y\}$ has a Λ-factor, $(c4)$ if $v(G) = 0 \bmod 3$ and G is either cubic or 4-connected, then $G - L$ has a Λ-factor, and $(c5)$ if G is cubic and E is a set of three edges in G, then $G - E$ has a Λ-factor if and only if the subgraph induced by E in G is not a claw and not a triangle.

Keywords: claw-free graph, cubic graph, Λ-packing, Λ-factor.

1 Introduction

We consider undirected graphs with no loops and no parallel edges. All notions and facts on graphs, that are used but not described here, can be found in [1, 2, 12].

Given graphs G and H, an H-*packing* of G is a subgraph of G whose every component is isomorphic to H. An H-*packing* P of G is called an H-*factor* if $V(P) = V(G)$. The H-*packing problem*, i.e. the problem of finding in G an H-packing, having the maximum number of vertices, turns out to be NP-hard if H is a connected graph with at least three vertices [3]. Let Λ denote a 3-vertex path. In particular, the Λ-packing problem is NP-hard. Moreover, this problem remains NP-hard even for cubic graphs [6].

Although the Λ-packing problem is NP-hard, i.e. possibly intractable in general, this problem turns out to be tractable for some natural classes of graphs (see, for example, **1.10** below). It would be also interesting to find polynomial-time algorithms that would provide a good approximation solution for the problem (e.g. **1.1** and **1.11** below). In each case the corresponding packing problem is polynomially solvable.

Let $v(G)$ and $\lambda(G)$ denote the number of vertices and the maximum number of disjoint 3–vertex paths in G, respectively. Obviously $\lambda(G) \le \lfloor v(G)/3 \rfloor$.

In [5, 11] we answered the following natural question:

How many disjoint 3-vertex paths must a cubic n-vertex graph have?

1.1 *If G is a cubic graph, then $\lambda(G) \ge \lceil v(G)/4 \rceil$. Moreover, there is a polynomial time algorithm for finding a Λ-packing having at least $\lceil v(G)/4 \rceil$ components.*

Obviously if every component of G is K_4, then $\lambda(G) = v(G)/4$. Therefore the bound in **1.1** is sharp.

Let $\mathcal{G}_2^3$ denote the set of graphs with each vertex of degree 2 or 3. In [5] we answered (in particular) the following question:

How many disjoint 3–vertex paths must an n-vertex graph from $\mathcal{G}_2^3$ have?

1.2 *Suppose that $G \in \mathcal{G}_2^3$ and G has no 5-vertex components. Then $\lambda(G) \ge v(G)/4$.*

Obviously **1.1** follows from **1.2** because if G is a cubic graph, then $G \in \mathcal{G}_2^3$ and G has no 5-vertex components.

In [5] we also gave a construction that allowed to prove the following:

1.3 *There are infinitely many connected graphs for which the bound in* **1.2** *is attained. Moreover, there are infinitely many subdivisions of cubic 3-connected graphs for which the bound in* **1.2** *is attained.*

The next interesting question is:

How many disjoint 3-vertex paths must a cubic connected graph have?

In [7] we proved the following. Let $\mathcal{C}_n$ denote the set of connected cubic graphs with n vertices.

1.4 *Let $\lambda_n = \min\{\lambda(G)/v(G) : G \in \mathcal{C}_n\}$. Then for some $c > 0$,*

$$\frac{3}{11}(1 - \frac{c}{n}) \le \lambda_n \le \frac{3}{11}(1 - \frac{1}{n^2}).$$

The similar question for cubic 2-connected graphs is still open:

1.5 Problem. *How many disjoint 3-vertex paths must a cubic 2-connected graph have?*

It is known that

1.6 *There are infinitely many 2-connected and cubic graphs G such that $\lambda(G) < \lfloor v(G)/3 \rfloor$.*

Some such graph sequences were constructed in [10] to provide 2-connected counterexamples to Reed's domination conjecture. Reed's conjecture claims that if G is a connected cubic graph, then $\gamma(G) \le \lceil v(G)/3 \rceil$, where $\gamma(G)$ is the dominating number of G (i.e. the size of a minimum vertex subset X in G such that every vertex in $G - X$ is adjacent to a vertex in X). In particular, a graph sequence $(R_k : k \ge 3)$ in [10] is such that each R_k is a cubic graph of connectivity two and $\gamma(G)/v(G) = \frac{1}{3} + \frac{1}{60}$. Obviously, $\gamma(G) \le v(G) - 2\lambda(G)$. Therefore $\lambda(R_k)/v(R_k) \le \frac{13}{40}$.

The questions arise whether the claim of **1.6** is true for cubic 2-connected graphs having some additional properties. For example,

1.7 Problem. *Is $\lambda(G) = \lfloor v(G)/3 \rfloor$ true for every 2-connected, cubic, bipartite, and planar graph ?*

In [8] we answered the question in **1.7** by giving a construction that provides infinitely many 2-connected, cubic, bipartite, and planar graphs such that $\lambda(G) < \lfloor v(G)/3 \rfloor$.

As to cubic 3-connected graphs, an old open question here is:

1.8 Problem. *Is the following claim true ?*
If G is a 3-connected and cubic graph, then $\lambda(G) = \lfloor v(G)/3 \rfloor$.

In [9] we discuss Problem **1.8** and show, in particular, that the claim in **1.8** is equivalent to some seemingly much stronger claims. Here are some results of this kind.

1.9 [9] *The following are equivalent for cubic 3-connected graphs G:*

$(z1)$ $v(G) = 0 \bmod 6 \Rightarrow$ *G has a Λ-factor,*

$(z2)$ $v(G) = 0 \bmod 6 \Rightarrow$ *for every $e \in E(G)$ there is a Λ-factor of G avoiding e,*

$(z3)$ $v(G) = 0 \bmod 6 \Rightarrow$ *for every $e \in E(G)$ there is a Λ-factor of G containing e,*

$(z4)$ $v(G) = 0 \bmod 6 \Rightarrow$ *$G - X$ has a Λ-factor for every $X \subseteq E(G)$, $|X| = 2$,*

$(z5)$ $v(G) = 0 \bmod 6 \Rightarrow$ *$G - L$ has a Λ-factor for every 3-vertex path L in G,*

$(t1)$ $v(G) = 2 \bmod 6 \Rightarrow$ *$G - \{x, y\}$ has a Λ-factor for every $xy \in E(G)$,*

$(f1)$ $v(G) = 4 \bmod 6 \Rightarrow$ *$G - x$ has a Λ-factor for every $x \in V(G)$,*

$(f2)$ $v(G) = 4 \bmod 6 \Rightarrow$ *$G - \{x, e\}$ has a Λ-factor for every $x \in V(G)$ and $e \in E(G)$.*

From **1.9** it follows that if the claim in Problem **1.8** is true, then Reed's domination conjecture is true for 3-connected cubic graphs.

There are some interesting results on the Λ-packing problem for so called claw-free graphs. A graph is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$ (which is called a *claw*).

A vertex x of a block B in G is called a *boundary vertex of B* if x belongs to another block of G. If B has exactly one boundary vertex, then B is called an *end-block* of G. Let $eb(G)$ denote the number of end-blocks of G.

1.10 [4] *Suppose that G is a claw-free graph and G is either 2-connected or connected with exactly two end-blocks. Then $\lambda(G) = \lfloor v(G)/3 \rfloor$.*

1.11 [4] *Suppose that G is a connected claw-free graph and $eb(G) \ge 2$. Then $\lambda(G) \ge \lfloor (v(G) - eb(G) + 2)/3 \rfloor$, and this lower bound is sharp.*

Obviously the claim in **1.10** about connected claw-free graphs with exactly two end-blocks follows from **1.11**.

In this paper (see Section 2) we give some more results on the Λ-packings in claw-free graphs. We show, in particular, the following:

$(c1)$ all claims in **1.9** except for $(z5)$ are true for 3-connected claw-free graphs and $(z5)$ is true for cubic, 2-connected, and claw-free graphs distinct from K_4 (see **2.15** below),

$(c2)$ if G is a 3-connected claw-free graph and $v(G) = 0 \bmod 3$, then for every edge e in G there exists a Λ-factor of G containing e (see **2.7**),

$(c3)$ if G is a 2-connected claw-free graph and $v(G) = 0 \bmod 3$, then for every edge e in G there exists a Λ-factor of G avoiding e, i.e. $G - e$ has a Λ-factor (see **2.8**),

$(c4)$ if G is a 2-connected claw-free graph and $v(G) = 1 \bmod 3$, then $G - x$ has a Λ-factor for every vertex x in G (see **2.13**),

$(c5)$ if G is a 3-connected claw-free graph and $v(G) = 2 \bmod 3$, then $G - \{x, y\}$ has a Λ-factor for every edge xy in G (see **2.3**),

$(c6)$ if G is a 3-connected, claw-free, and cubic graph with $v(G) \ge 6$ or a 4-connected claw-free graph, then for every 3-vertex path L in G there exists a Λ-factor containing L, i.e. $G - L$ has a Λ-factor (see **2.1** and **2.6**),

$(c7)$ if G is a cubic, 3-connected, and claw-free graph with $v(G) \ge 6$ and E is a set of three edges in G, then $G - E$ has a Λ-factor if and only if the subgraph induced by E in G is not a claw and not a triangle (see **2.10**),

$(c8)$ if G is a 3-connected claw-free graph, $v(G) = 1 \bmod 3$, $x \in V(G)$, and $e \in E(G)$, then $G - \{x, e\}$ has a Λ-factor (see **2.14**).

2 Main results

Theorems **1.10** and **1.11** above describe some properties of maximum Λ-packings in claw-free graphs. In this section we establish some more properties of Λ-packings in claw-free graphs.

Let G be a graph and B be a block of G, and so B is either 2-connected or consists of two vertices and one edge. As above, a vertex x of B is called a *boundary vertex of* B if x belongs to another block of G, and an *inner vertex of* B, otherwise. If B has exactly one boundary vertex, then B is called an *end-block* of G.

Let F be a graph, $x \in V(F)$, and $X = \{x_1, x_2, x_3\}$ be the set of vertices in F adjacent to x. Let T be a triangle, $V(T) = \{t_1, t_2, t_3\}$, and $V(F) \cap V(T) = \emptyset$. Let $G = (F - x) \cup T \cup \{x_i t_i : i \in \{1, 2, 3\}$. We say that G *is obtained from* F *by replacing a vertex* x *by a triangle*. Let F^Δ denote the graph obtained from a cubic graph F by replacing each vertex of F by a triangle. Obviously, F^Δ is claw-free, every vertex belongs to exactly one triangle, and every edge belongs to at most one triangle in F^Δ.

2.1 *Let G' be a cubic 2-connected graph and G be the graph obtained from G' by replacing each vertex v of G' by a triangle Δ_v. Let L be a 3-vertex path in G. Then*

(a) *$G - L$ has a Λ-factor.*

Moreover,

$(a1)$ *if L induces a triangle in G, then G has a Λ-factor R containing L and such that each component of R induces a triangle*

$(a2)$ *if L does not induce a triangle in G, then G has a Λ-factor R containing L and such that no component of R induces a triangle, and*

$(a3)$ *if L does not induce a triangle in G, then G has a Λ-factor, containing L and a component that induces a triangle.*

Proof Let $L = xzz_1$. Let E' be the set of edges in G that belong to no triangle. Obviously, there is a natural bijection $\alpha : E(G') \to E'$. Since each vertex of G belongs to exactly one triangle, we can assume that xz belongs to a triangle $T = xzs$.

(**p1**) Suppose that L induces a triangle in G, and so $s = z_1$. Obviously the union of all triangles in G contains a Λ-factor, say P, of G and $L \subset P$. Therefore claim $(a1)$ is true.

(**p2**) Now suppose that L does not induce a triangle in G, and so $s \ne z_1$. Let $\bar{s} = ss_1$ and $\bar{z} = zz_1$ be the edges of G not belonging to T, and therefore belonging to no triangles in G. Hence $\bar{s} = \alpha(\bar{s}')$ and $\bar{z} = \alpha(\bar{z}')$, where $\bar{s}' = s's'_1$ and $\bar{z}' = z'z'_1$ are edges in G', and $s' = z'$. Since every vertex in G belongs to exactly one triangle, clearly $s_1 \ne z_1$.

(**p2.1**) We prove $(a2)$. By using Tutte's criterion for a graph to have a perfect matching, it is easy to prove the following:

Claim. *If A is a cubic 2-connected graph, then for every 3-vertex path J of A there exists a 2-factor of A containing J.*

By the above Claim, G' has a 2-factor F' containing 3-vertex path $S' = s'_1 s' z'_1$. Let C' be the (cycle) component of F' containing S'. If Q' is a (cycle) component of F', then let Q be the subgraph of G, induced by the edge subset $\{\alpha(e) : e \in E(Q')\} \cup \{E(\Delta_v) : v \in V(Q')\}$. Obviously $v(Q) = 0 \bmod 3$ and Q has a (unique) Hamiltonian cycle $H(Q)$. Also the union F of all Q's is a spanning subgraph of G and each Q is a component of F. Moreover, if C is the component in F, corresponding to C', then $L \subset H(C)$. Therefore each $H(Q)$ has a Λ-factor $P(Q)$, such that no component of $P(Q)$ induces a triangle, and $H(C)$ has a (unique) Λ-factor $P(C)$, such that $L \subset P(C)$ and no component of $P(C)$ induces a triangle. The union of all these Λ-factors is a Λ-factor P of G containing L and such that no component of P induces a triangle. Therefore $(a2)$ holds.

(**p2.2**) Now we prove $(a3)$. Since G' is 2-connected and cubic, there is a cycle C' in G' such that $V(C') \ne V(G')$ and C' contains $S' = s'_1 s' z'_1$. Let, as above, C be the subgraph of G, induced by the edge subset $\{\alpha(e) : e \in E(C')\} \cup \{E(\Delta_v) : v \in V(C')\}$. Obviously, $v(C) = 0 \bmod 3$, C has a (unique) Hamiltonian cycle H, and $L \subset H$. Therefore H has a (unique) Λ-factor $P(C)$ containing L. Since $V(C') \ne V(G')$, we have $V(G' - C') \ne \emptyset$.

Therefore $G-C$ has a triangle. Moreover, every vertex v in $G-C$ belongs to a unique triangle Δ_v, and therefore as in (**p1**), $G-C$ has a Λ-factor Q whose every component induces a triangle in $G-C$. Then $P(C)\cup Q$ is a required a Λ-factor in G. $\square$

Theorem **2.1** is not true for a cubic, 2-connected, and claw-free graph F with an edge xy belonging to two triangles T_i with $V(T_i)=\{x,y,z_i\}$ because $L=z_1xz_2$ is a 3-vertex path in F and y is an isolated vertex in $F-L$.

2.2 *Suppose that G is a 2-connected claw-free graph, $v(G)=2 \bmod 3$, and $x\in V(G)$. Then there exist at least two edges xz_1 and xz_2 in G such that each $G-xz_i$ is connected and has a Λ -factor.*

Proof (uses **1.10**). We need the following simple facts.

CLAIM 1. *Let G be a 2-connected graph and, $x\in V(G)$. Then there exist at least two edges xs_1 and xs_2 in G such that each $G-xs_i$ is connected.*

CLAIM 2. *Let G be a claw-free graph, B is a 2-connected block of G, and x is a boundary vertex of B. Then $B-x$ is either 2-connected or has exactly one edge.*

By CLAIM 1, G has an edge xy such that $G-\{x,y\}$ is connected. If $G-\{x,s\}$ for every $xs\in E(G)$, then by CLAIM 1, we are done. Therefore we assume that $G-\{x,y\}$ is connected but has no Λ - factor.

Then by **1.10**, $G-\{x,y\}$ has at least three end-blocks, say B_i, $i\in\{1,\dots,k\}$, $k\geq 3$. Let b'_i be the boundary vertex of B_i. Let V_i be the set of vertices in $\{x,y\}$ adjacent to the interior of B_i and $\mathcal{B}_v$ be the set of the end-bocks in $G-\{x,y\}$ whose interior is adjacent to $v\in\{x,y\}$. Since G is 2-connected, each $|V_i|\geq 1$. Since G is claw-free, each $|\mathcal{B}_v|\leq 2$. Since $k\geq 3$, $|\mathcal{B}_v|=2$ for some $v\in\{x,y\}$, say for $v=x$ and $\mathcal{B}_x=\{B_1,B_2\}$. Let $xb_i\in E(G)$, where b_i is an interior vertex of B_i, $i\in\{1,2\}$, and let $xb_j\in E(G)$, where b_j is an interior vertex of B_j, $j\geq 3$. Since G is claw-free, $\{x,y,b_1,b_2\}$ does not induce a claw in G. Therefore $yb_2\in E(G)$. If $k\geq 4$, then $\{y,b_2,b_3,b_4\}$ induces a claw in G, a contradiction. Thus $k=3$ and $\mathcal{B}_y=\{B_2,B_3\}$.

Suppose that $v(B_s)=0 \bmod 3$ for some $s\in\{1,2,3\}$. Then B_s is 2-connected. Since B_s is claw-free, by **1.10**, B_s has a Λ-factor, say P. Since $v(G)=2 \bmod 3$, we have $v(G-\{x,y,B_s\})=0 \bmod 3$. By CLAIM 2, $G-\{x,y,B_s\}$ is claw-free, connected and has at most two end-blocks. Then by **1.10**, $G-\{x,y,B_s\}$ has Λ-factor, say Q. Therefore $P\cup Q$ is a Λ-factor of $G-\{x,y\}$, a contradiction.

Suppose that $v(B_r)=1 \bmod 3$ for some $r\in\{1,2,3\}$. Then B_r is 2-connected. Obviously $v(B_r-b'_r)=0 \bmod 3$ and claw-free. By CLAIM 2, $B_r-b'_r$ is 2-connected. Then by **1.10**, B_r has a Λ-factor, say P. Obviously $G-\{x,y,B_r-b'_r\}$ is claw-free, connected and has at most two end-blocks. Then by **1.10**, $G-\{x,y,B_s\}$ has Λ-factor, say Q. Therefore $P\cup Q$ is a Λ-factor of $G-\{x,y\}$, a contradiction.

Now suppose that $v(B_i)=2 \bmod 3$ for every $i\in\{1,2,3\}$. By CLAIM 2, either $B_i-\{b_i,b'_i\}$ is 2-connected or $v(B_i-\{b_i,b'_i\})=0$ for every $i\in\{1,2,3\}$. In both cases

by the arguments similar to that above, $G - \{x, b_i\}$ has a Λ-factor for $i \in \{1, 2\}$ and $G - \{y, b_i\}$ has a Λ-factor for $i \in \{2, 3\}$. $\square$

From **2.2** we have for 3-connected claw-free graphs the following stronger result with a simpler proof.

2.3 *Suppose that G is a 3-connected claw-free graph, $v(G) = 2 \bmod 3$, and $xy \in E(G)$. Then $G - \{x, y\}$ has a Λ-factor.*

Proof (uses **1.10**). Let $G' = G - \{x, y\}$. Since G is 3-connected, G' is connected. By **1.10**, it suffices to prove that G' has at most two end-blocks. Suppose, on the contrary, that G' has at least three end-blocks. Let B_i, $i \in \{1, 2, 3\}$, be some three blocks of G'. Since G is 3-connected, for every block B_i and every vertex $v \in \{x, y\}$ there is an edge vb_i, where b_i is an inner vertex of B_i. Then $\{v, b_1, b_2, b_3\}$ induces a claw in G, a contradiction. $\square$

As we have seen in the proof of **2.2**, the claim of **2.3** is not true for claw-free graphs of connectivity two.

2.4 *Suppose that G is a 3-connected claw-free graph, $v(G) = 0 \bmod 3$, and $xy \in E(G)$. Then there exist at least two 3-vertex paths L_1 and L_2 in G centered at y, containing xy, and such that each $G - L_i$ is connected and has a Λ-factor.*

Proof (uses **1.10**). We need the following simple fact.

CLAIM 1. *Let G be a 3-connected graph, $x \in V(G)$, and $xy \in E(G)$. Then there exist two 3-vertex paths L_1 and L_2 in G centered at y, containing xy, and such that each $G - L_i$ is connected.*

By CLAIM 1, G has a 3-vertex path $L = xyz$ such that $G - L$ is connected. If every such 3-vertex path belongs to a Λ-factor of G, then by CLAIM 1, we are done. Therefore we assume that $G - L$ is connected but has no Λ - factor. Then by **1.10**, $G - L$ has at least three end-blocks, say B_i, $i \in \{1, \ldots, k\}$, $k \geq 3$. Let b'_i be the boundary vertex of B_i. Let V_i be the set of vertices in L adjacent to inner vertices of B_i and $\mathcal{B}_v$ be the set of the end-bocks in $G - L$ having an inner vertex adjacent to v in $V(L)$. Since G is 3-connected, each $|V_i| \geq 2$. Since G is claw-free, each $|\mathcal{B}_v| \leq 2$. It follows that $k = 3$, each $|V_i| = 2$, each $|\mathcal{B}_v| = 2$, as well as all V_i's are different and all $\mathcal{B}_v$'s are different. Let $s^1 = z$, $s^2 = x$, $s^3 = y$, and $S = \{s^1, s^2, s^3\}$. We can assume that $V_i = S - s^i$, $i \in \{1, 2, 3\}$. Then for every vertex $s^j \in V_i$ there is has a vertex b^j_i in $B_i - b'_i$ adjacent to s^j, where $\{b^j_i : s^j \in V_i\}$ has exactly one vertex if and only if $B_i - b'_i$ has exactly one vertex. Let $L_i = s^2 s^3 b_i$, where $b_i = b^3_i$.

By **1.10**, it surfices to show that each $G - L_i$ is connected and has at most two end-blocks.

Let $i = 1$. If $B_1 - b_1$ is 2-connected, then $B_1 - b_1$ and $G - L_1 - (B_1 - b'_1)$ are the two end-blocks of $G - L_1$ and we are done. If $B_1 - b_1$ is empty, then $G - L_1$ is 2-connected.

So we assume that $B_1 - b_1$ is not empty and not 2-connected. Then $B_1 - b_1$ is connected and has exactly two end-blocks, say C_1 and C_2. Let c'_i be the boundary vertex of C_i in $B_1 - b_1$. Since G is 3-connected, each $C_i - c'_i$ has a vertex adjacent to $\{s^2, s^3\}$. We can assume that a vertex c_1 in $C_1 - c'_1$ is adjacent to s^2. If there exists a vertex c_2 in $C_2 - c'_2$ adjacent to s^2, then $\{s^2, b^2_3, c_1, c_2\}$ induces a claw in G, a contradiction. So suppose that no vertex in $C_2 - c'_2$ is adjacent to s^2. Then there is a vertex c_2 in $C_2 - c'_2$ adjacent to s^3. Then $\{s^2, s^3, b^3_2, c_2\}$ induces a claw in G, a contradiction.

Now let $i = 2$. If $B_2 - b_2$ is 2-connected, then B_1 and $G - L_2 - (B_1 - b'_1)$ are the two end-blocks of $G - L_2$ and we are done. If $B_1 - b_1$ is empty, then $G - L_2$ has two end-blocks, namely B_1 and the subgraph of G induced by $B_3 \cup s^1$. So we assume that $B_2 - b_2$ is not empty and not 2-connected. Then $B_2 - b_2$ is connected and has exactly two end-blocks, say D_1 and D_2. Let d'_i be the boundary vertex of D_i in $B_2 - b_2$. Since G is 3-connected, each $D_i - d'_i$ has a vertex adjacent to $\{s^1, s^3\}$. We can assume that a vertex d_1 in $D_1 - d'_1$ is adjacent to s^3. If there exists a vertex d_2 in $D_2 - d'_2$ adjacent to s^3, then $\{s^3, d_1, d_2, b^3_1\}$ induces a claw in G, a contradiction. So suppose that no vertex in $D_2 - d'_2$ is adjacent to s^3. Then there is a vertex d_2 in $D_2 - d'_2$ adjacent to s^1. Then $\{s^1, s^3, b^1_3, d_2\}$ induces a claw in G, a contradiction. $\square$

From the proof of **2.4** we have, in particular:

2.5 *Suppose that G is a 3-connected claw-free graph. If L is a 3-vertex path and the center vertex of L has degree 3 in G, then $G - L$ is connected and has a Λ-factor in G.*

Obviously, **2.1** (a) follows from **2.5**.

From the proof of **2.4** we also have:

2.6 *Suppose that G is a 4-connected claw-free graph. Then $G - L$ is connected and has a Λ-factor for every 3-vertex path L in G.*

The claim of **2.6** may not be true for a claw-free graph of connectivity 3 if they are not cubic. A graph obtained obtained from a claw by replacing its vertex of degree 3 by a triangle is called a *net*. Let N be a net with the three leaves v_1, v_2, and v_3, T a triangle with $V(T) = \{t_1, t_2, t_3\}$, and let N and T be disjoint. Let $H = N \cup T \cup \{v_i t_j : i, j \in \{1, 2, 3\}, i \neq j\}$. Then H is a 3-connected claw-free graph, $v(H) = 9$, each $d(t_i, H) = 4$, $d(x, H) = 3$ for every $x \in V(H - T)$, and $H - T = N$ has no Λ-factor. If L is a 3-vertex path in T, then $H - L = H - T$, and so $H - L$ has no Λ-factor. There are infinitely many pairs (G, L) such that G is a 3-connected, claw-free, and non-cubic graph, $v(G) = 0 \bmod 3$, L is a 3-vertex path in G, and $G - L$ has no Λ-factor. By **2.9**, such a pair can be obtained from the above pair (H, L) by replacing N by any graph A with three leaves from the class $\mathcal{A}$ (defined below before **2.9**) provided $v(A) = 0 \bmod 3$.

From **2.4** we have, in particular:

2.7 *Suppose that G is a 3-connected claw-free graph and $e \in E(G)$. Then*

$(a1)$ *there exists a Λ-factor in G containing e and*

$(a2)$ *there exists a Λ-factor in G avoiding e, i.e. $G - e$ has a Λ-factor.*

The following examples show that condition "*G is a 3-connected graph*" in **2.7** is essential for claim $(a1)$. Let R be the graph obtained from two disjoint cycles A and B by adding a new vertex z, and the set of new edges $\{a_i z, b_i z : i \in \{1,2\}\}$, where $a = a_1 a_2 \in E(A)$ and $b = b_1 b_2 \in E(B)$. It is easy to see that Q is a claw-free graph of connectivity one. Furthermore, if $v(A) = 1 \bmod 3$ and $v(B) = 1 \bmod 3$, then $v(Q) = 0 \bmod 3$ and Q has no Λ-factor containing edge $e \in \{a, b\}$. Similarly, let Q be the graph obtained from two disjoint cycles A and B by adding two new vertices z_1 and z_2, a new edge $e = z_1 z_2$, and the set of new edges $\{a_i z_j, b_i z_j : i, j \in \{1,2\}\}$, where $a_1 a_2 \in E(A)$ and $b_1 b_2 \in E(B)$. It is easy to see that Q is a claw-free graph of connectivity two. Furthermore, if $v(A) = 2 \bmod 3$ and $v(B) = 2 \bmod 3$, then $v(Q) = 0 \bmod 3$ and Q has no Λ-factor containing edge e.

As to claim $(a2)$ in **2.7**, it turns out that this claim is also true for 2-connected claw-free graphs.

2.8 *Suppose that G is a 2-connected claw-free graph, $v(G) = 0 \bmod 3$, and $e \in E(G)$. Then $G - e$ has a Λ-factor.*

Proof A graph H is called *minimal 2-connected* if H is 2-connected but $H - u$ is not 2-connected for every $u \in E(H)$. A *frame* of a graph G is a minimal 2-connected spanning subgraph of G. Clearly, every 2-connected graph has a frame. In [4] we describe Procedure 1 that provides an ear-assembly A of a special frame of a 2-connected claw-free graph. In particular, the last ear of A contains a Λ-packing P such that $G - P$ is also 2-connected claw-free graph. We modify Procedure 1 by replacing the first step of this procedure "*Find a longest cycle G_0 in G*" by "*Find a longest cycle G'_0 among all cycles C in G such that edge e either belongs to C or is a chord of C*". Since G is 2-connected, G has a cycle containing e. Therefore a cycle G'_0 does exist. Then the resulting Procedure $\mathcal{P}$ provides an ear-assembly of a frame of G with the property that the last ear of this frame has a Λ-packing Q such that $e \notin E(Q)$ and $G - Q$ is a 2-connected claw-free graph that may contain e.

We can use Procedure $\mathcal{P}$ to prove our claim by induction on $v(G)$. If G is a cycle containing e, then our claim is obviously true. Procedure $\mathcal{P}$ mentioned above guarantees the existence of a Λ-packing Q such that Q avoids e and $G - Q$ is a 2-connected claw-free graph that may contain e. Obviously $v(G - Q) = 0 \bmod 3$ and $v(G - Q) < v(G)$. By the induction hypothesis, $G - Q$ has a Λ-factor R avoiding e. Then $Q \cup R$ is a Λ-factor of G avoiding edge e. $\square$

We need the following fact interesting in itself. Let $\mathcal{A}$ denote the set of graphs A with the following properties:

($c1$) A is connected,

($c2$) every vertex in A has degree at most 3,

($c3$) every vertex in A of degree 2 or 3 belongs to exactly one triangle, and

($c4$) A has exactly three vertices of degree 1 which we call the *leaves* of A.

2.9 *If $A \in \mathcal{A}$, then A has no Λ-factor.*

Proof Let $A \in \mathcal{A}$. If $v(A) \neq 0 \bmod 3$, then our claim is clearly true. So we assume that $v(A) = 0 \bmod 3$. We prove our claim by induction on $v(G)$. The smallest graph in $\mathcal{A}$ is a net N with $v(N) = 6$ and our claim is obviously true for N. So let $v(A) \geq 9$. Suppose, on the contrary, that A has a Λ-factor P. Let v be a leaf of A and vx the edge incident to v. Since P is a Λ-factor in A, it has a component $L = vxy$, and so $P - L$ is a Λ-factor in $A - L$ and $d(x, A) \geq 2$. By property ($c3$), x belongs to a unique triangle xyz in A and $d(x, a) = 3$, and so $s \in \{y, z\}$. If $d(z, A) = 2$, then z is an isolated vertex in $A - L$, and so P is not a Λ-factor in A, a contradiction. Therefore by ($c2$), $d(z, A) = 3$. Therefore $A - L$ satisfies ($c2$), ($c3$), and ($c4$).

Suppose that $G - L$ is not connected and that the three leaves do not belong to a common component. Then $A - L$ has a component C with $v(C) \neq 0 \bmod 3$, and so $A - L$ has no Λ-factor, a contradiction.

Now suppose that $A - L$ has a component C containing all three leaves of $A - L$. Then $C \in \mathcal{A}$ and $v(C) < v(A)$. By the induction hypothesis, C has no Λ-factor. Therefore $A - L$ also has no Λ-factor, a contradiction. $\square$

Given $E \subseteq E(G)$, let $\dot{E}$ denote the subgraph of G induced by E.

2.10 *Suppose that G is a cubic 2-connected graph and that every vertex in G belongs to exactly one triangle (and so G is claw-free), i.e. $G = F^{\Delta}$, where F is a cubic 2-connected graph. Let $E \subset E(G)$ and $|E| = 3$. Then the following are equivalent:*

(g) *$G - E$ has no Λ-factor and*

(e) *$\dot{E}$ satisfies one of the following conditions:*

($e1$) *$\dot{E}$ is a claw,*

($e2$) *$\dot{E}$ is a triangle,*

($e3$) *$\dot{E}$ has exactly two components, the 2-edge component $\dot{E}_2$ belongs to a triangle in G, the 1-edge component $\dot{E}_1$ belongs to no triangle in G, and $G - E$ is not connected, and*

($e4$) *$\dot{E}$ has exactly two components, in G the 2-edge component $\dot{E}_2$ belongs to a triangle, say T, the 1-edge component $\dot{E}_1$ also belongs to a triangle, say D, and $\dot{E}_1$, $\dot{E}_2$ belong to different component of $G - \{d, t\}$, where d is the edge incident to the vertex of $D - \dot{E}_1$ and t is the edge in $G - E$ incident to the isolated vertex of $T - E$.*

Proof (uses **1.10**, **2.1**(a), and **2.9**). Let $X, Y \subset E(G)$ such that X meets no triangle in G, each edge in Y belongs to a triangle in G, and no triangle in G has more than one

edge from Y, and so $X \cap Y = \emptyset$. We will use the following simple observation.

CLAIM. *$G - X - Y$ has a Λ-factor P such that every component of P induces a triangle in G and if an edge y from Y is in a triangle T, then $T - y$ is a component of P.*

Let $E = \{a, b, c\}$. By the above CLAIM, we can assume that edges a and b belong to the same triangle T.

(**p1**) We prove $(e) \Rightarrow (g)$.

Suppose that $\dot{E}$ satisfies $(e1)$, i.e. $\dot{E}$ is a claw. Then $G - E$ has an isolated vertex and therefore has no Λ-factor.

Suppose that $\dot{E}$ satisfies $(e2)$, i.e. $\dot{E}$ is a triangle. Then $G - E \in \mathcal{A}$. By **2.9**, $G - E$ has no Λ-factor.

Suppose that $\dot{E}$ satisfies $(e3)$, i.e. $G - E$ is not connected and $\dot{E}$ has exactly two components $\dot{E}_2$ and $\dot{E}_1$ induced by $\{a, b\}$ and c, respectively, where $\dot{E}_2$ belongs to the triangle T but $\dot{E}_1$ belongs to no triangle in G. Then t is the dangling edge in $G - E$. Let S be the component in $G - E$ containing edge t. Then every vertex in S distinct from the leaf incident to t belongs to exactly one triangle. Therefore $v(S) = 1 \bmod 3$. Thus $G - E$ has no Λ-factor.

Now suppose that $\dot{E}$ satisfies $(e4)$. By $(e4)$, t is the edge in $G - E$ incident to z. Suppose, on the contrary, that $G - E$ has a Λ-factor, say P. Since $G - \{d, t\}$ is not connected, $G - E - E(D)$ is also not connected. Obviously the component of $G - E - E(D)$ containing z belongs to $\mathcal{A}$. Therefore by **2.9**, $G - E - E(D)$ has no Λ-factor. Thus P has a 3-vertex path L containing exactly one edge in D adjacent to edge d. Now if C is a component of $G - E - L$, then $v(C) \neq 0 \bmod 3$. Therefore $G - E$ has no Λ-factor, a contradiction.

(**p2**) Now we prove $(g) \Rightarrow (e)$. Namely, we assume that $\dot{E}$ does not satisfy (e) and we want to show that in this case $G - E$ has a Λ-factor. Let u be the edge of T distinct from a and b.

Suppose that $\dot{E}$ is connected, and so $\dot{E}$ is a 3-edge path. Let V be a 3-vertex path in G containing u and avoiding E. Then $G - V$ has no edges from E, and so $G - V = G - E - V$. By **2.1**(a), $G - V$ has a Λ-factor.

Now suppose that $\dot{E}$ is not connected, and so $\dot{E}$ has exactly two components induced by $\{a, b\}$ and by c, respectively. Since $\dot{E}$ does not satisfy (e), $\dot{E}$ is not a claw and not a triangle, and so $u, t \notin E$.

(**p2.1**) Suppose that c belongs to no triangle in G. Since $\dot{E}$ does not satisfy (e), $G - E$ is connected. Clearly, $G - E$ is claw-free. Also $G - E$ has exactly two end-blocks and the block of one edge t is one of them. By **1.10**, $G - E$ has a Λ-factor.

(**p2.2**) Now suppose that c belongs to a triangle D in G. Then $D \neq T$. Let $G' = G - D - \{a, b\}$. Then G' is claw-free and has no edges from E.

Suppose that G' is connected. Then as in (**p2.1**), G' has exactly two end-blocks. By **1.10**, G' has a Λ-factor, say P. Let $L = D - c$, and so L is a 3-vertex path. Then

$P \cup \{L\}$ is a Λ-factor in $G - E$.

Now suppose that G' is not connected. Let C be the component of G' containing edge t and q the edge connecting C and D. Let L be a 3-vertex path in G containing q and an edge in $D - c$. It is sufficient to show that $G - E - L$ has a Λ-factor. Obviously $G - E - L$ is claw-free. Let Q be a component of $G - E - L$. Since $\dot{E}$ does not satisfy $(e4)$, $v(Q) = 0 \bmod 3$ and Q has exactly two end-blocks. By **1.10**, Q has a Λ-factor. Therefore $G - E - L$ also has a Λ-factor. $\square$

From **2.10** we have, in particular:

2.11 *Suppose that $G = F^{\Delta}$, where F is a cubic 2-connected graph (and so G is claw-free). Let $E \subset E(G)$ and $|E| = 2$. Then $G - E$ has a Λ-factor.*

From **2.10** we also have:

2.12 *Suppose that G is a 3-connected claw-free graph. Let $E \subset E(G)$ and $|E| = 3$. Then $G - E$ has a Λ-factor if and only if $\dot{E}$ is not a claw and not a traingle.*

2.13 *Suppose that G is a 2-connected claw-free graph, $v(G) = 1 \bmod 3$ and $x \in V(G)$. Then $G - x$ has a Λ-factor.*

Proof (uses **1.10**). Let $x \in V(G)$. Since $v(G) = 1 \bmod 3$, clearly $v(G-x) = 0 \bmod 3$. Since G is 2-connected, $G - x$ is connected. Since G is claw-free, $G - x$ is claw-free and has at most two end-blocks. By **1.10**, $G - x$ has a Λ-factor. $\square$

2.14 *Suppose that G is a 3-connected claw-free graph, $v(G) = 1 \bmod 3$, $x \in V(G)$ and $e \in E(G)$. Then $G - \{x, e\}$ has a Λ-factor.*

Proof (uses **2.8** and **2.13**). Since G is 3-connected, $G - x$ is a 2-connected claw-free graph. Since $v(G) = 1 \bmod 3$, we have $v(G - x) = 0 \bmod 3$. By **2.13**, $G - x$ has a Λ-factor P. If $e \notin E(G - x)$, then P is a Λ-factor of $G - \{x, e\}$. If $e \in E(G - x)$, then by **2.8**, $G - \{x, e\}$ has a Λ-factor. $\square$

Obviously, the claim in **2.14** may not be true for a claw-free graph of connectivity 2.

From **2.1**, **2.3**, **2.7**, **2.11**, **2.13**, and **2.14** we have, in particular:

2.15 *All claims in* **1.9** *except for $(z5)$ are true for 3-connected claw-free graphs and $(z5)$ is true for cubic, 2-connected graphs such that every vertex belongs to exactly one triangle.*

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